IB Linear Algebra – Example Sheet 2

Michaelmas 2024 hb470@cam.ac.uk

1. Write down the three types of elementary matrices and find their inverses. Use elementary matrices to find the inverse of
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

2. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathbb{M}_{m\times r}(\mathbb{F})$ and $C \in \mathbb{M}_{r\times n}(\mathbb{F})$.

   Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

3. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & -A \\ B & 0 \end{pmatrix} \text{ can be transformed into } D = \begin{pmatrix} I & 0 \\ B & AB \end{pmatrix}$$
by elementary column operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det(AB) = \det(A)\det(B)$.

4. Let $\alpha : V \to V$ be a linear map on a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
   (i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
   (ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

5. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then
$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$
   Show that the rank of the adjugate matrix is $\mathrm{rk}(\mathrm{adj}\,A) = \begin{cases} n & \text{if } \mathrm{rk}(A) = n \\ 1 & \text{if } \mathrm{rk}(A) = n-1 \\ 0 & \text{if } \mathrm{rk}(A) \leq n-2. \end{cases}$

   Do (i)-(iii) hold if A is singular? *[Hint: for (i) consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]*

6. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
   (a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
   (b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
   (c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
   (d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

7. For $A \in \mathbb{M}_{n\times m}(\mathbb{F})$ and $B \in \mathbb{M}_{m\times n}(\mathbb{F})$, let $\tau_A(B)$ denote $\mathrm{tr}(AB)$. Show that, for each fixed A, $\tau_A\colon \mathbb{M}_{m\times n}(\mathbb{F}) \to \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathbb{M}_{n\times m}(\mathbb{F}) \to \mathbb{M}_{m\times n}(\mathbb{F})^*$.

8. (a) Suppose that $f \in \mathbb{M}_{n\times n}(\mathbb{R})^*$ is such that $f(AB) = f(BA)$ for all $A, B \in \mathbb{M}_{n\times n}(\mathbb{R})$ and $f(I) = n$. Show that f is the trace functional, i.e. $f(A) = \mathrm{tr}A$ for all $A \in \mathbb{M}_{n\times n}(\mathbb{R})$.

   (b) Now let V be a non-zero finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = id_V$.

   (c) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α and β of V such that $\alpha\beta - \beta\alpha = id_V$.

9. (a) Let $a_0, ..., a_n$ be distinct real numbers, and let

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}.$$

Show that $\det(A) \neq 0$.

(b) Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbf{R}$ define $e_x \in P_n^*$ by $e_x(p) = p(x)$. By considering the standard basis $(1, t, \ldots, t^n)$ for P_n, use (a) to show that $\{e_0, ..., e_n\}$ is linearly independent and hence forms a basis for P_n^*.

(c) Identify the basis of P_n to which $(e_0, ..., e_n)$ is dual.

10. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
(a) The map D defined by $D(p)(t) = p'(t)$.
(b) The map S defined by $S(p)(t) = p(t^2)$.
(c) The composite DS.
(d) The composite SD.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

11. Let V be a finite dimensional vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that g is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$. What if V is infinite dimensional?